MM Optimization Algorithms
Homework 4

Iterative Refinement

Let f be a quadratic function, i.e., $f(x) = (1/2)x^{\mathrm{T}}Ax - b^{\mathrm{T}}x$, where A is an element of the set $\mathbb{S}^n_{++}$ of symmetric *positive definite* $n \times n$ matrices. It is straightforward to see that that the *unique* minimizer of f is given by the solution of the linear system $Ax = b$.

1. By using the *data files* [1] of A and b given in `A.mat` and `b.mat`, respectively, compute numerically the minimizer $x^{\mathtt{direct}}$ of f given by $x^{\mathtt{direct}} = A^{-1}b$. What is the the numerical error e of the computed minimizer defined as $e(x^{\mathtt{direct}}) = \|b - Ax^{\mathtt{direct}}\|$?

2. Show that

$$\begin{aligned} \mathtt{prox}_{\mu f}(x) &= (A + (1/\mu)I)^{-1}\,(b + (1/\mu)x) \\ &= x + [(1/\mu)I + A]^{-1}(b - Ax). \end{aligned} \tag{1}$$

   where I is the identity matrix and μ is a positive scalar.

3. Let $x^{(0)} = [1 \cdots 1]^{\mathrm{T}} \in \mathbb{R}^n$. Implement the proximal minimization algorithm

$$x^{(n+1)} = \mathtt{prox}_{\mu f}(x^{(n)})$$

   with $\mu = 1000$ for 20 iterations, i.e., $n = 0, \ldots, 19$. Plot the graph of error versus iterations, i.e., $e(x^{(k)})$ for $k = 0, \ldots, 20$.

4. Compare the numerical errors $e(x^{(20)})$ and $e(x^{\mathtt{direct}})$. What may be the reasons for their significant difference, if any?

[1] The data files are stored in MATLAB mat form.