A Brief on Optimization

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Optimization

What is Optimization

- in its abstraction, optimization is to minimize a function f_0 over a set $\mathcal{F}$
- e.g., LASSO
  - $f_0(x) = ||y - Ax||_2$: the objective function
  - $\mathcal{F} = \{x \mid ||x||_1 \leq r\}$: the feasible set
- $x \in \mathbb{R}^n$ is called the decision variable
- $y \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and $r \in \mathbb{R}$ are given

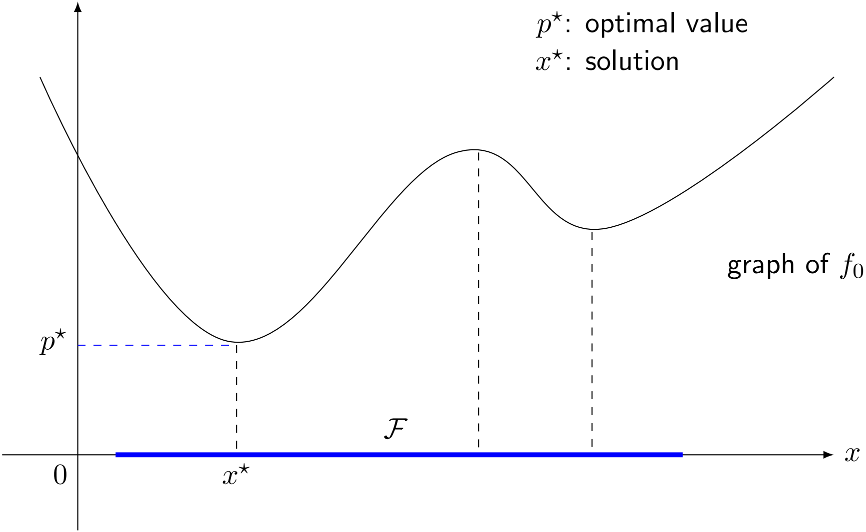
Problem Formulation

- a general formulation of an optimization problem

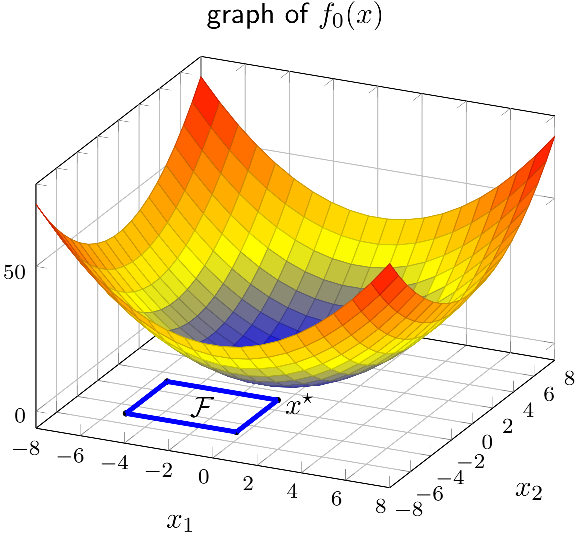
$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & x \in \mathcal{F} \end{array}$$

- the decision variable is x
- f_0 and $\mathcal{F}$ depend on the application
- f_0 encodes what we want to optimize
- $\mathcal{F}$ encodes the underlying constraints

Geometric Interpretation



Geometric Interpretation



Solving the Problem

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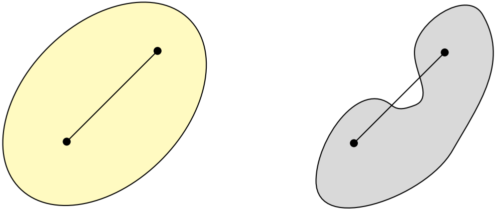
- can all optimization problems be efficiently solved ?
  - No !
  - there is a general class of problems which are efficiently solved
  - called convex optimization problems

Convex Optimization

Convex Optimization Problems

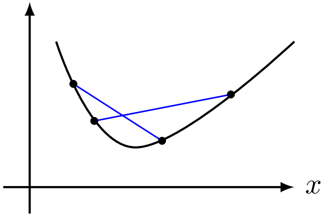
- what makes an optimization problem a convex optimization problem ?
- determined by the characteristics of
  - $\mathcal{F}$
  - f_0

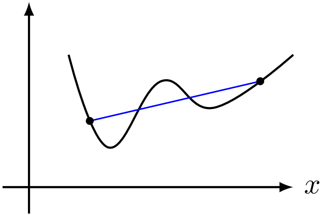
Characteristics of $\mathcal{F}$



- the line segment through any distinct points in $\mathcal{F}$ should lie in $\mathcal{F}$
- condition above holds $\rightarrow$ $\mathcal{F}$ is said to be convex
- set in the left is convex, set in the right is not convex

Characteristics of $f_0(x)$





- domain of f_0 should be convex
- the chord through any distinct points on the graph of f_0 should lie above the graph
- conditions above hold $\rightarrow$ f_0 is said to be convex
- function in the left is convex, the one in the right is not convex

Convex Optimization Problems

- thus, roughly speaking, if
  - $\mathcal{F}$ is convex
  - f_0 is convex
- then the underlying optimization problem is convex

Importance of Convexity

Importance of Convexity

- a fundamental property of convex optimization problems:
  - any local solution is also a global solution
- optimality criterions can be specified
- by using the optimality criterions, closed form solutions are achieved in many circumstances
  - e.g., water-filling algorithm

Importance of Convexity

- by using the optimality criterions, efficient algorithms can be designed to find the solution
- e.g.,
  - unconstrained optimization: gradient descent algorithm, steepest descent algorithm, Newton's method
  - equality constrained optimization: Newton's method with equality constraints
  - constrained optimization: interior-point method

Importance of Convexity

- iterative algorithms can be designed to find the solution, even in the case of large scale problems
  - ADMM
  - proixmal algorithms
- iterative algorithms can be designed to for decentralized the solution methods
  - decomposition methods
  - ADMM
  - proixmal algorithms

Importance of Convexity

- the algorithms aforementioned are gracefully implemented in many software packages
- a popular one is
  - `CVX`, matlab software for convex optimization
- for a complete list, you may refer to the stanford website, `http://web.stanford.edu/~boyd/software.html`

Many Application Domains

- machine learning
- information retrieval
- engineering design
- economics
- finance
- management

Why Convex Optimization

- many application domains
- massive data sets $\Longrightarrow$ optimization is a crucial component of the emerging field of data science
- many problems in statistics and machine learning $\Longrightarrow$ in the framework of convex optimization
- can be considered as 'must know' area for engineers

An Application: LASSO

Image Compression

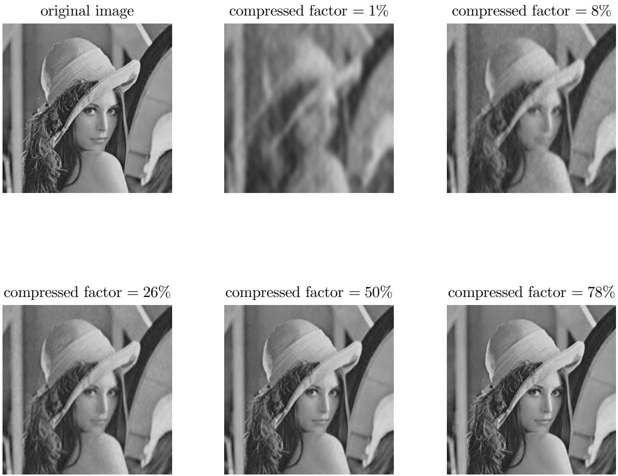
- consider the 225×225 gray-scale [1] image

original image

[1] Each pixel in the image is represented by an integer value y_i in the range $[0, 255]$.

Image Compression

- least absolute shrinkage and selection operator (LASSO) to compress the image



LASSO

- how the LASSO is designed ?

LASSO

- how the LASSO is designed ?
- the underlying mathematics
  - OPTIMIZATION !

LASSO as an Optimization Problem

LASSO: Problem Formulation

- so roughly speaking, the problem is to
  - 'approximate' y by Ax
  - by letting x to be sparse
  - sparsity of x is the key to image compression
- technically, the optimization problem is

$$\begin{array}{ll} \underset{x}{\text{minimize}} & ||y - Ax||_2 \\ \text{subject to} & ||x||_1 \leq r, \end{array}$$

where $r \in \mathbb{R}$ is a parameter to decide the level of compression

LASSO: Problem Formulation

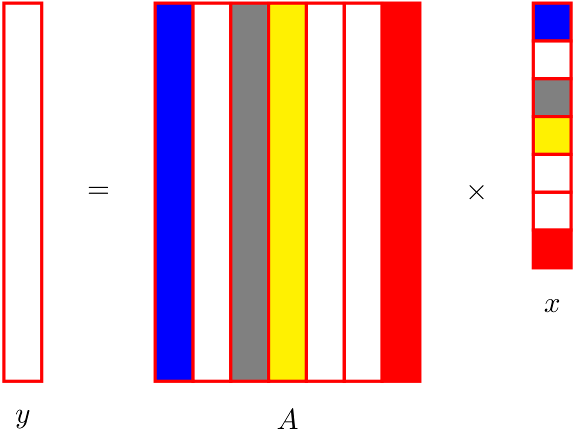
- recall, we have a 225×225 gray-scale image
- it is represented by a vector $y = [y_1, y_2, \ldots, y_n]^\mathsf{T} \in \mathbb{R}^n$
  - note that $n = 50625$
- for an *appropriate* dictionary matrix, one can represent y as

$$y = Ax,$$

- A is a $n \times n$ matrix, x is a n-vector
- simply y is obtained by linearly combining columns of A, the recipe is x !

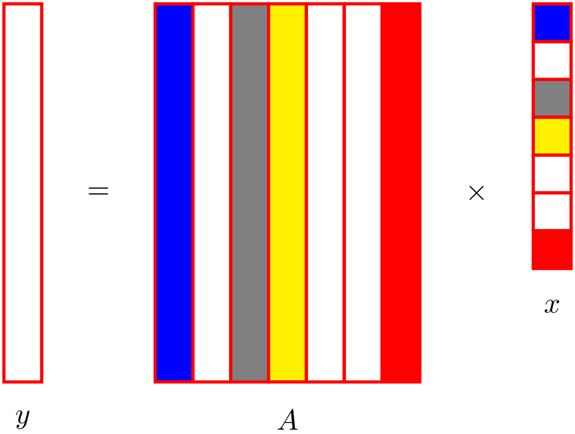
LASSO: Problem Formulation

- linearly combining columns of A to form y, the recipe is x



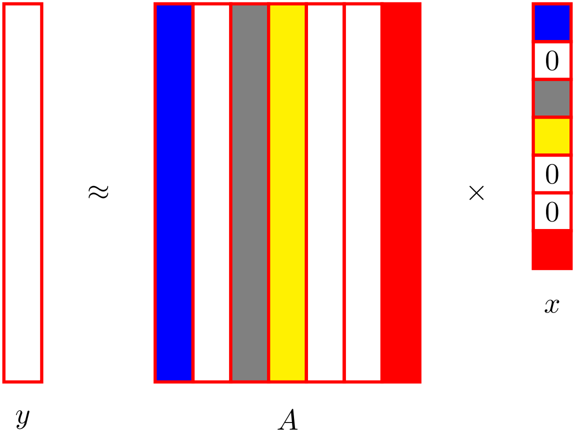
LASSO: Problem Formulation

- do we really need all the columns of A to reconstruct the image y ? YES !, if no losses are allowed



LASSO: Problem Formulation

- however, we may get an approximation to y if columns of insignificant contribution are eliminated



Some References

Some Key References

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